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# QED<sub>2</sub> in Curved Backgrounds

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## Abstract

Here we discuss the two dimensional quantum electrodynamics in curved space-time, especially in the background of some black holes. We first show the existence of some new quantum mechanical solution which has interesting properties. Then for some special black holes we discuss the fermion-black hole scattering problem. The issue of confinement is intimately connected with these solutions and we also comment on this in this background. Finally, the entanglement entropy and the Hawking radiation are also discussed in this background from a slightly different viewpoint.

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# 1 Introduction

The Schwinger model [1], in curved two-dimensional space-time, has been studied over the years [2] and it has been speculated that the qualitative behavior of the model should not change in the presence of gravity [3]. The comparison can be made with the finite temperature QED<sub>2</sub> [4] where no non-trivial phase appears and the model stays only in one phase which is the screened Coloumb phase that exists at the zero temperature. However, a rigorous evidence about this was missing for the curved backgrounds. In this paper we develop a general formalism of the model for an arbitrary curved background and finally gather evidences about the phases for some particular background. What we observe is that the phase structure remains unaltered at least for this special background and the model may exist in two phases, namely in screened Coulomb and the unconfining phases, just as it was in the flat case. This observation has crucial impact on the study of fermion-black hole scattering problem which was done in an earlier paper [5]. Earlier we neglected the gravitational interactions and found that the model itself can not avoid the problem of information loss. In this paper we shall take up the same problem but now not neglecting the gravity. However, as the model doesn't show any qualitative change the conclusions are obvious - the model still supports the information loss. The important lesson that we learn from this is that this problem can possibly be avoided only if we incorporate the quantum gravitational effects.

In recent years another important area of interest has been the study of matter fields in the black hole backgrounds. It provides a lot of insights into the problem of black hole entropy, the study of Hawking radiation and all these raised more issues about the quantum theory of gravity. As the spectrum of QED<sub>2</sub> contains only a scalar field the study of QED<sub>2</sub> in black hole backgrounds boils down to the study of scalar fields. In this

paper we study this problem, namely we first calculate the entanglement entropy of a scalar field in the particular black hole background which we considered in section 1 and discuss the Hawking radiation of this black hole. We show that the whole analysis can be done with sufficient simplicity.

## 2 QED<sub>2</sub> in curved background

QED<sub>2</sub> or the Schwinger model in curved space-time is described by the Lagrangian density [1]

$$\mathcal{L} = -\frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} + i \bar{\psi} \gamma^\mu(x) D_\mu \psi. \quad (2.1)$$

where the indices  $\mu, \nu \dots$  etc. refer to the curved background and take the values 0,1. Other notations are standard. To introduce the fermions we need to go to a locally flat space-time with which the correspondences are established via the zweibeins. The zweibeins satisfy the relations

$$\begin{aligned} e^{\mu a}(x) \gamma_a &= \gamma^\mu(x), & e^{\mu a}(x) e_\mu^b(x) &= \eta^{ab}, \\ e^{\mu a}(x) e_{\nu a}(x) &= \delta_\nu^\mu, & e^{\mu a}(x) e_{\mu b}(x) &= \delta_b^a. \end{aligned} \quad (2.2)$$

The indices  $a, b \dots$  etc. are denoting the flat space-time, the flat space indices are raised and lowered by the metric  $\eta_{ab}$  and the curved space indices are raised and lowered by  $g_{\mu\nu}$ . The field strength being antisymmetric in the space-time indices continues to have the same form  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . However, the gauge covariant derivative acting on the fermions is of the form,  $D_\mu \psi = (\nabla_\mu - ie A_\mu) \psi$ , where  $\nabla_\mu = \partial_\mu + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab}$ .  $\sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]$  are the standard Lorentz transformation generators and  $\omega_\mu^{ab}$  are the spin connections. We shall be working in the gauge  $\nabla_\mu e^{\nu a} = 0$ , which fixes the spin connections completely in terms of the zweibeins  $\omega_{\mu ab} = \frac{1}{2} [e_a^\nu (\partial_\mu e_{\nu b} - \partial_\nu e_{\mu b}) + \frac{1}{2} e_a^\rho e_b^\sigma (\partial_\sigma e_{\rho c}) e_\mu^c - (a \leftrightarrow b)]$ . Throughout we shall

be using the following notations and conventions: for flat space  $\eta^{ab} = \text{diag}(1, -1)$ ,  $\gamma^5 = \gamma^0\gamma^1$ ,  $(\gamma^0)^2 = -(\gamma^1)^2 = 1$  and  $\epsilon_{01} = +1$ . For the curved background  $\sqrt{-g}\gamma^\mu\epsilon_{\mu\nu} = \gamma^5\gamma_\nu$  and  $\widetilde{\nabla}_\mu = \sqrt{-g}\epsilon_{\mu\nu}\nabla^\nu$ .

The partition function is given by

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS}, \quad S = \int d^2x \sqrt{-g} \mathcal{L}. \quad (2.3)$$

The effective action is defined by the following functional of the abelian gauge field  $A_\mu$

$$e^{i\Gamma[A]} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^2x \sqrt{-g} \bar{\psi} i \not{D} \psi}. \quad (2.4)$$

Now in two dimensions we can always set

$$A_\mu = -\frac{\sqrt{\pi}}{e} (\widetilde{\nabla}_\mu \sigma + \nabla_\mu \tilde{\eta}) \quad (2.5)$$

where,  $\sigma$  and  $\tilde{\eta}$  are scalar fields. So the field strength is given by  $F_{\mu\nu} = \frac{\sqrt{\pi}}{e} \epsilon_{\mu\nu} \sqrt{-g} \square \sigma$  where,  $\square \sigma = \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \sigma)$ . The Dirac operator is given by

$$\not{D} = \not{\nabla} + i\sqrt{\pi} \not{\nabla} \tilde{\eta} + i\sqrt{\pi} \gamma^5 \not{\nabla} \sigma. \quad (2.6)$$

It is easy to see that the transformations

$$\begin{aligned} \psi &\rightarrow e^{i\sqrt{\pi}(\tilde{\eta} - \gamma^5 \sigma)} \psi, \\ \bar{\psi} &\rightarrow \bar{\psi} e^{-i\sqrt{\pi}(\tilde{\eta} + \gamma^5 \sigma)}, \end{aligned} \quad (2.7)$$

decouple the gauge field from the fermions and the classical action becomes free, *i.e.*,

$$\bar{\psi} i \not{D} \psi \rightarrow \bar{\psi} i \not{\partial} \psi. \quad (2.8)$$

However, we should proceed through infinitesimal steps. For the time being it is sufficient to consider only the chiral transformations since the effective action is known to be invariant under the standard gauge transformations. But quantum mechanically the chiral

symmetry becomes anomalous because the fermionic measure does not remain invariant under those transformations. To see this explicitly let us make the following infinitesimal chiral redefinition of the fermionic variables,

$$\begin{aligned}\psi &\rightarrow \psi_\delta = (1 - i\sqrt{\pi}\gamma^5\delta\sigma)\psi, \\ \bar{\psi} &\rightarrow \bar{\psi}_\delta = \bar{\psi}(1 - i\sqrt{\pi}\gamma^5\delta\sigma)\end{aligned}\tag{2.9}$$

leading to

$$\begin{aligned}\bar{\psi}\not{D}\psi &= \bar{\psi}_\delta[\not{D} + i\sqrt{\pi}\gamma^5\nabla(\sigma - \delta\sigma)]\psi_\delta \\ &= \bar{\psi}_\delta\not{D}\psi_\delta - i\sqrt{\pi}\bar{\psi}_\delta\gamma^5\nabla(\delta\sigma)\psi_\delta.\end{aligned}\tag{2.10}$$

So the effective action becomes

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS} \\ &= \int \mathcal{D}\bar{\psi}_\delta\mathcal{D}\psi_\delta e^{iS_\delta} \\ &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS} [1 + i\sqrt{\pi} \int d^2x \sqrt{-g} \bar{\psi}\gamma^5\nabla(\delta\sigma)\psi] \\ &= \mathcal{Z} + i\sqrt{\pi} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS} \int d^2x \sqrt{-g} \delta\sigma \nabla_\mu J_5^\mu.\end{aligned}\tag{2.11}$$

where,  $J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ . This apparently shows that  $\nabla_\mu J_5^\mu = 0$  which in turn implies the presence of chiral invariance. However, it is well known that  $\mathcal{D}\bar{\psi}_\delta\mathcal{D}\psi_\delta \neq \mathcal{D}\bar{\psi}\mathcal{D}\psi$ . To calculate explicitly the Jacobian we proceed as follows.

We first analytically continue the space-time to the Euclidean domain, i.e.  $x^0 \rightarrow -ix^4$ ,  $\gamma^0 \rightarrow i\gamma^4$ ,  $D^0 \rightarrow iD^4$ . Then we choose a set of complete orthonormal functions  $\{\Phi_n(x)\}$  satisfying

$$\int d^2x \sqrt{g} \Phi_n^\dagger \Phi_m = \delta_{nm} \quad \sum_n \sqrt{g} \Phi_n(x) \Phi_n^\dagger(y) = \delta^2(x - y).\tag{2.12}$$

The fermionic fields can be expanded in terms of these functions as  $\psi(x) = \sum_n a_n \Phi_n(x)$  and  $\bar{\psi}(x) = \sum_n \Phi_n^\dagger(x) b_n$ , where  $a_n$  and  $b_n$ 's are Grassmann numbers. The fermionic measures are accordingly expressed as  $\mathcal{D}\psi = \prod_n da_n$  and  $\mathcal{D}\bar{\psi} = \prod_n db_n$ . Therefore, under the chiral transformation  $\psi \rightarrow \psi_\delta = \sum_n a_n^\delta \Phi_n = \sum_{nm} C_{nm} a_m \Phi_n(x)$  the measure changes as  $\mathcal{D}\psi_\delta = \prod da_n^\delta = \frac{1}{\det C} \prod_n da_n = \frac{1}{\det C} \mathcal{D}\psi$ . So the Jacobian is the determinant of the matrix

$$C_{nm} = \delta_{nm} - i\sqrt{\pi} \int d^2x \sqrt{g} \delta\sigma(x) \Phi_n^\dagger(x) \gamma^5 \Phi_m(x). \quad (2.13)$$

So

$$\begin{aligned} \det C &= \exp(\text{Tr} \ln C) \\ &= \exp \left[ -i\sqrt{\pi} \int d^2x \sqrt{g} \delta\sigma(x) \sum_n \Phi_n^\dagger(x) \gamma^5 \Phi_n(x) \right]. \end{aligned} \quad (2.14)$$

The entire measure is just the square of this

$$\mathcal{D}\bar{\psi}_\delta \mathcal{D}\psi_\delta = \frac{1}{(\det C)^2} \mathcal{D}\bar{\psi} \mathcal{D}\psi = \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[ 2i\sqrt{\pi} \int d^2x \sqrt{g} \delta\sigma(x) \sum_n \Phi_n^\dagger(x) \gamma^5 \Phi_n(x) \right]. \quad (2.15)$$

The sum in the exponent is divergent since arbitrary higher modes go into the summation. We shall adopt here the regularization procedure demonstrated by Fujikawa [6] by letting the higher modes to damp exponentially. For this let us consider a gauge invariant Dirac like operator whose eigenfunctions are to be identified with these complete set of orthonormal functions. The operator must be chosen to be gauge invariant in order to ensure that the gauge invariance of the measure is not destroyed in the process of regularization. The operator also need to be Dirac like since otherwise the trace over the Dirac gamma matrices will produce a trivial zero result. So in order to avoid the triviality and restore the gauge invariance one is sufficiently restricted in the choice of the regularizing operator. One natural choice, which Fujikawa himself employed [6] has been

the Euclidean Dirac operator of the action itself. However, after that many people [7, 8] realized that this is not that sacred and one such explicit varied choice was suggested in [9]. It seems to be rather unique general possibility obeying the above constraints if one combines with them the requirement that the linearity of the theory is to be maintained. As the theory in a curved background doesn't have a global translation invariance the regularizing operator can more generally be taken similar to [5]

$$\mathcal{D}_r \Phi_n = \lambda \Phi_n, \quad \mathcal{D}_r = \gamma_\mu D_\mu^r = \gamma_\mu (\nabla_\mu - ie A_\mu^r) \quad (2.16)$$

where  $A_\mu^r$  is the regularizing background which is taken to be  $A_\mu^r = A_\mu - \nabla_\nu(\varphi(x)F_{\mu\nu})$ . The scalar field  $\varphi$  is breaking the global translation invariance. We have taken  $\varphi$  to be a function of the space only.

Now usually in flat space-time we use the plane wave representation of the basis  $\{\Phi_n\}$  to evaluate the trace. But in the curved background due to the loss of global translation invariance there is no global representation of the momentum. But this can be achieved in the Riemann normal coordinates  $\xi^\mu = (x - y)^\mu$  and the use of these coordinates is reasonable since we will be working in the small space-time regions to calculate the trace. So we proceed as follows

$$\begin{aligned} \left[ \int d^2x \sqrt{g} \delta\sigma(x) \Phi_n^\dagger \gamma^5 \Phi_n \right]_{\text{reg}} &= \lim_{M^2 \rightarrow \infty} \sum_n \int d^2x \sqrt{g} \delta\sigma(x) \Phi_n^\dagger(x) \gamma^5 \Phi_n(x) e^{-\lambda_n^2/M^2} \\ &= \lim_{M^2 \rightarrow \infty} \sum_n \int d^2x \sqrt{g} \delta\sigma(x) \Phi_n^\dagger(x) \gamma^5 e^{-\mathcal{D}_r^2/M^2} \Phi_n(x) \\ &= \lim_{\substack{M^2 \rightarrow \infty \\ x \rightarrow y}} \int d^2x \delta\sigma(x) \text{Tr} \gamma^5 e^{-\mathcal{D}_r^2/M^2} \delta^2(x - y) \\ &= \lim_{\substack{M^2 \rightarrow \infty \\ \xi \rightarrow 0}} \int d^2x \delta\sigma(x) \int \frac{d^2p}{(2\pi)^2} \text{Tr} \gamma^5 e^{-\mathcal{D}_r^2/M^2} e^{-ip \cdot \xi} \quad (2.17) \end{aligned}$$

It is necessary to calculate the explicit form of the operator  $\mathcal{D}_r^2$  and then the trace of the gamma matrices and the limits are to be taken carefully. First using the formula

$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb} - (\mu \leftrightarrow \nu)$  we get  $\mathcal{D}_r^2 = D_\mu^r D_\mu^r + \frac{1}{12} R + \frac{e}{2\sqrt{g}} \epsilon_{\mu\nu} \gamma^5 F_{\mu\nu}^r$ , where  $F_{\mu\nu}^r = \partial_\mu A_\nu^r - \partial_\nu A_\mu^r$ . Using these

$$\left[ \int d^2x \sqrt{g} \delta\sigma(x) \Phi_n^\dagger \gamma^5 \Phi_n \right]_{\text{reg}} = -\frac{e}{4\pi} \int d^2x \delta\sigma(x) \epsilon_{\mu\nu} F_{\mu\nu}^r. \quad (2.18)$$

So, the Jacobian is

$$\mathcal{D}\bar{\psi}_\delta \mathcal{D}\psi_\delta = \mathcal{D}\bar{\psi} \mathcal{D}\psi \left[ 1 - \frac{ie\sqrt{\pi}}{2\pi} \int d^2x \delta\sigma(x) \epsilon_{\mu\nu} F_{\mu\nu}^r \right]. \quad (2.19)$$

Thus there is a chiral anomaly given by

$$\nabla_\mu J_5^\mu = \frac{1}{\sqrt{-g}} \frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}^r. \quad (2.20)$$

Using the expression  $\frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}^r = -\sqrt{-g} \frac{1}{\sqrt{\pi}} \square(\sigma + \varphi \square \sigma)$

$$\nabla_\mu J_5^\mu = -\frac{1}{\sqrt{\pi}} \square(\sigma + \varphi \square \sigma). \quad (2.21)$$

Let us now go to calculate the effective action/bosonized action. By a small change of variables, both in  $\sigma$  and  $\tilde{\eta}$ , we find that the measure is offering a Jacobian only under the change of the chiral variables. Thus we arrived at the equation

$$e^{i\Gamma[\sigma, \tilde{\eta}]} = (\det C_{\text{reg}})^2 e^{i\Gamma[\sigma - \delta\sigma, \tilde{\eta} - \delta\tilde{\eta}]}. \quad (2.22)$$

Making the Taylor expansion of the effective functional we get

$$\delta\Gamma[\sigma] = \int d^2x \sqrt{-g} \delta\sigma(x) \square(\sigma + \varphi \square \sigma), \quad \frac{\delta\Gamma}{\delta\tilde{\eta}} = 0 \quad (2.23)$$

leading to

$$\Gamma[\sigma] = \frac{1}{2} \int d^2x \sqrt{-g} [\sigma \square \sigma + \varphi \square \sigma \square \sigma]. \quad (2.24)$$

Re-expressing this in terms of the gauge fields

$$\Gamma[A] = \int d^2x \sqrt{-g} \left[ \frac{e^2}{2\pi} \tilde{\nabla} \cdot A \frac{1}{\square} \tilde{\nabla} \cdot A - \frac{e^2 \varphi}{4\pi} g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} \right]. \quad (2.25)$$



The local form of the resulting bosonized action can be obtained by introducing an auxiliary field  $\Sigma$

$$S_B = \int d^2x \sqrt{-g} \left[ -\frac{1}{4} \left( 1 + \frac{e^2 \varphi}{\pi} \right) g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} + \frac{e^2}{2\pi} A^2 + \frac{1}{2} \nabla_\mu \Sigma \nabla^\mu \Sigma - \frac{e}{\sqrt{\pi}} A^\mu \nabla_\mu \Sigma \right]. \quad (2.26)$$

Then one can do the standard constraint analysis to calculate the Hamiltonian. First, the canonical momenta have to be defined. The momenta corresponding to  $A_0$ ,  $A_1$  and  $\Sigma$  are respectively

$$\begin{aligned} \Pi^0 &= \frac{\delta S_B}{\delta \nabla_0 A_0} = 0 \\ \Pi^1 &= \frac{\delta S_B}{\delta \nabla_0 A_1} = \frac{1}{\sqrt{-g}} \left( 1 + \frac{e^2 \varphi}{\pi} \right) (\nabla_0 A_1 - \nabla_1 A_0) \\ \Pi^\Sigma &= \frac{\delta S_B}{\delta \nabla_0 \Sigma} = \sqrt{-g} \left( \nabla^0 \Sigma - \frac{e}{\sqrt{\pi}} A^0 \right). \end{aligned} \quad (2.27)$$

The first of these equations is recognized to be a constraint. Using all these equations, we obtain the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \Pi^1 \nabla_0 A_1 + \Pi^\Sigma \nabla_0 \Sigma - \mathcal{L}_B \\ &= \frac{\sqrt{-g} (\Pi^1)^2}{2(1 + e^2 \varphi / \pi)} + \Pi^1 \nabla_1 A_0 - \frac{1}{2} \sqrt{-g} g^{11} (\nabla_1 \Sigma)^2 + \frac{e}{\sqrt{\pi}} \sqrt{-g} A^1 \nabla_1 \Sigma - \frac{e^2}{2\pi} \sqrt{-g} A^2 \\ &\quad + \frac{1}{2g^{00} \sqrt{-g}} \left[ \Pi^\Sigma + \sqrt{-g} \left( \frac{e}{\sqrt{\pi}} A^0 - g^{01} \nabla_1 \Sigma \right) \right]^2. \end{aligned} \quad (2.28)$$

The consistency that the first constraint equation be invariant under time evolution by this Hamiltonian requires a secondary constraint which is the Gauss law

$$G \equiv \nabla_1 \Pi^1 - \frac{e}{\sqrt{\pi}} \Pi^\Sigma = 0. \quad (2.29)$$

There are no further constraints, and it can be checked that the Poisson brackets of  $G$  and  $\Pi^0$  vanish, so that the constraints are *first class*. This is natural, as we have taken

care to maintain gauge invariance in the effective action. As usual, then, we have to fix a gauge to remove gauge degrees of freedom. It is convenient here to consider the physical gauge conditions

$$\Sigma = A_0 = 0. \quad (2.30)$$

In the present gauge, the Hamiltonian simplifies to

$$\mathcal{H} = \frac{\sqrt{-g} (\Pi^1)^2}{2(1 + e^2\varphi/\pi)} + \frac{1}{g^{00}} \nabla_1 \Pi^1 g^{01} A_1 + \frac{e^2}{2\pi} \frac{A_1^2}{g^{00} \sqrt{-g}} + \frac{\pi}{e^2} \frac{(\nabla_1 \Pi^1)^2}{2 g^{00} \sqrt{-g}}. \quad (2.31)$$

Note that  $\mathcal{H}$  is preserved in time since we allow  $\varphi$  to depend on space only. Now it is interesting to notice that this can be brought to a Hamiltonian of a “free field” in a curved background. We should keep in mind that a “free field” Hamiltonian in a gravitational field is no longer really a free theory as the particles can interact gravitationally. If we start from the Lagrangian of a scalar field in a curved space-time

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} (\nabla^\mu \Phi \nabla_\mu \Phi - M^2 \Phi^2), \quad (2.32)$$

the Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2g^{00}\sqrt{-g}} (\Pi^\Phi - \sqrt{-g} g^{01} \nabla_1 \Phi)^2 - \frac{1}{2} \sqrt{-g} g^{11} (\nabla_1 \Phi)^2 + \frac{1}{2} \sqrt{-g} M^2 \Phi^2 \quad (2.33)$$

where  $\Pi^\Phi = \sqrt{-g} \nabla^0 \Phi$  is the canonically conjugate momentum of  $\Phi$ . Comparing (2.31) and (2.33) our Hamiltonian may be converted to the familiar “free field” form by the redefinitions

$$\Phi = \frac{\sqrt{\pi}}{e} \Pi^1, \quad \Pi^\Phi = -\frac{e}{\sqrt{\pi}} A_1. \quad (2.34)$$

This shows that the physical spectrum of the model contains just a massive boson with mass  $M = e/\sqrt{\pi + e^2\varphi}$ .

Let us investigate the nature of the force mediated by the gauge field of this theory between two quarks. First, in the presence of two static external quarks ( $q\bar{q}$ -pair) of charge  $Q$  at  $\pm L/2$ , the charge density is modified to

$$\begin{aligned} J_Q^0(t, x^1) &= \frac{Q}{e} \sqrt{\frac{g_{00}}{g}} \left[ \delta(x^1 - \frac{L}{2}) - \delta(x^1 + \frac{L}{2}) \right] + J^0 \\ &= J^0 - \frac{1}{\sqrt{\pi}} \nabla_1 \chi, \end{aligned} \quad (2.35)$$

where,

$$\chi = \frac{Q\sqrt{\pi}}{e} \theta(x^1 + \frac{L}{2}) \theta(\frac{L}{2} - x^1) \sqrt{\frac{g_{00}}{g}}. \quad (2.36)$$

Remembering that  $eJ_\mu = \delta\Gamma[A]/\delta A^\mu$  the Lagrangian density in the presence of these external quarks can be written as

$$\mathcal{L}_Q = \mathcal{L}_B - \frac{e}{\sqrt{\pi}} \sqrt{-g} \widetilde{\nabla} \cdot A \chi. \quad (2.37)$$

The momenta corresponding to  $\chi, A_0, A_1$  and  $\Sigma$  are respectively

$$\begin{aligned} \Pi_Q^\chi &= \frac{\partial \mathcal{L}_Q}{\partial \nabla_0 \chi} = 0 \\ \Pi_Q^0 &= \frac{\partial \mathcal{L}_Q}{\partial \nabla_0 A_0} = 0 \\ \Pi_Q^1 &= \frac{\partial \mathcal{L}_Q}{\partial \nabla_0 A_1} = \Pi^1 - \frac{e}{\sqrt{\pi}} \chi \\ \Pi_Q^\Sigma &= \frac{\partial \mathcal{L}_Q}{\partial \nabla_0 \Sigma} = \Pi^\Sigma. \end{aligned} \quad (2.38)$$

The first two of these equations are recognized to be primary constraints. Using all these equations, we obtain the Hamiltonian

$$\begin{aligned} \mathcal{H}_Q &= \Pi_Q^1 \nabla_0 A_1 + \Pi_Q^\Sigma \nabla_0 \Sigma - \mathcal{L}_Q \\ &= \mathcal{H} - \frac{e}{\sqrt{\pi}} \nabla_1 A_0 \chi. \end{aligned} \quad (2.39)$$

The consistency that the primary constraint equations be invariant under time evolution by this Hamiltonian requires secondary constraints which are

$$\begin{aligned} G_1 &\equiv \nabla_1 \Pi_Q^1 - \frac{e}{\sqrt{\pi}} \Pi^\Sigma = 0 \\ G_2 &\equiv \frac{e}{\sqrt{\pi}} \nabla_1 A_0 = 0. \end{aligned} \quad (2.40)$$

There are no further constraints, and it can be checked that the mutual Poisson brackets of the constraints with one another vanish, so that the constraints are *first class*. This is natural, as we have taken care to maintain gauge invariance in the effective action. As usual, then, we have to fix a gauge to remove gauge degrees of freedom. It is convenient here to consider the physical gauge conditions

$$\Sigma = A_0 = 0. \quad (2.41)$$

In the present gauge, the Hamiltonian simplifies to

$$\mathcal{H}_Q = \frac{\sqrt{-g} (\Pi^1)^2}{2(1 + e^2 \varphi / \pi)} + \frac{1}{g^{00}} \nabla_1 \Pi_Q^1 g^{01} A_1 + \frac{e^2}{2\pi} \frac{A_1^2}{g^{00} \sqrt{-g}} + \frac{\pi}{e^2} \frac{(\nabla_1 \Pi_Q^1)^2}{2g^{00} \sqrt{-g}}. \quad (2.42)$$

Now it is interesting to notice that this can also be brought to a Hamiltonian almost similar to a “free one” by the following redefinition of fields

$$\tilde{\Pi}^\Phi = \Pi^\Phi, \quad \tilde{\Phi} = \Phi + \chi = \frac{\sqrt{\pi}}{e} \Pi_Q^1 \quad (2.43)$$

leading to

$$\mathcal{H}_Q = \frac{(\tilde{\Pi}^\Phi)^2}{2g^{00} \sqrt{-g}} - \frac{1}{g^{00}} \tilde{\Pi}^\Phi g^{01} \nabla_1 \tilde{\Phi}^2 + \frac{(\nabla_1 \tilde{\Phi})^2}{2g^{00} \sqrt{-g}} + \frac{1}{2} \sqrt{-g} \frac{e^2}{\pi + e^2 \varphi} (\tilde{\Phi} - \chi)^2 \quad (2.44)$$

Then the potential between the quark-antiquark pair would be the difference between the ground state energies of these two Hamiltonians  $\mathcal{H}_Q$  and  $\mathcal{H}$  and this can be calculated

because both the Hamiltonians are still quadratic in the momenta. The straightforward path-integral evaluation gives

$$V(L) = E_Q - E = \frac{1}{2} \int dx^1 \sqrt{-g} \left[ \frac{e^2}{\pi + e^2 \varphi} \chi^2 + \left( \frac{e^2}{\pi + e^2 \varphi} \chi \right) \frac{1}{\nabla^1 \nabla_1 - \frac{e^2}{\pi + e^2 \varphi}} \left( \chi \frac{e^2}{\pi + e^2 \varphi} \right) \right]. \quad (2.45)$$

If we compare this expression with the potential we obtain in the flat background we see that there is great similarity between them. For our purpose let us take the expression given in [9]

$$V(L) = \frac{1}{2} \int dx^1 \left[ \frac{e^2}{\pi + e^2 a} \chi^2 + \left( \frac{e^2}{\pi + e^2 a} \chi \right) \frac{1}{\partial_1^2 - \frac{e^2}{\pi + e^2 a}} \left( \chi \frac{e^2}{\pi + e^2 a} \right) \right] \quad (2.46)$$

where in  $\chi$  here we have to put the flat Minkowski metric and also  $\varphi = a$  is a constant parameter. It is almost impossible to calculate the nature of the potential for an arbitrary background. So we consider some special case. In the next section we shall be considering a particular example of a  $(1+1)$ -dimensional black hole which is a solution of string theory. Let us concentrate on that solution here. As the  $x^1$ -coordinate runs from  $-\infty$  to  $+\infty$  it is to be identified with the tortoise coordinate of the black hole. For details of the solution see the next section. In the tortoise coordinate the metric looks like ( we put  $x^1 \equiv \sigma$  in the next section)

$$ds^2 = \left(1 - \frac{2M}{r}\right) (-dt^2 + (dx^1)^2) \quad (2.47)$$

where  $(1 - 2M/r) = (1 + \frac{M}{\lambda} e^{-2\lambda x^1})^{-1} = \sqrt{-g}$ . Also  $\nabla^1 \nabla_1 = (1 + \frac{M}{\lambda} e^{-2\lambda x^1})(\nabla_1)^2$  and  $\chi = (Q\sqrt{\pi}/e) \theta(x^1 + L/2) \theta(L/2 - x^1) (1 + \frac{M}{\lambda} e^{-2\lambda x^1})$ . Now the task is to evaluate the integrals. For the time being let us take  $\varphi = \text{const.}$  which we deed in [9] and try to calculate  $V(L)$  for large  $L$ . Then the first integral becomes clearly the same as the first one in (2.46) After a little thought the second one will also turned out to be the same

as the second one in (2.46). To see this explicitly just make a change of variable from  $x^1$  to  $x^1/L$ . So in the limit  $L \rightarrow \infty$  the potential  $V(L)$  doesn't really alter at all even in this non-trivial background. It has been argued a long time ago that finite curvature is like finite temperature and as the Schwinger model doesn't show any non-trivial phase at finite temperature its behavior is not expected to change in the presence of gravity. We proved here this conjecture explicitly at least for a non-trivial background.

We can now make some comments about the phases of the model. Since we have a free space dependent parameter in our solution we can see the nature of the spectrum and quark interactions for various forms of this function. Again for simplicity let us put the function  $\varphi$  to be a constant. In that case the model can be found only in two phases similar to the flat case, namely i) the constant  $\rightarrow 0$ : Then the mass of the boson is  $m = e/\sqrt{\pi}$  and the potential  $V(L) \sim \text{const.}$  for large  $L$ . So this is the familiar screened Coloumb phase and ii) the constant  $\rightarrow \infty$ : Then the mass  $m = 0$  and also  $V(L) = 0, \forall L$ . So the quarks become essentially free and the mass-less boson can be interpreted as a mass-less fermion [10]. Thus the fermions get liberated into the spectrum and this is an unconfining phase.

One interesting application of this model has already been discussed in [5], i.e. in the problem of fermion - black hole scattering [11]. There we have ignored the gravitational degrees of freedom and eventually the background was set to be flat [12, 5]. Now we shall study the same problem in an arbitrary curved background. For that we have to include the dilaton and the relevant action is essentially the string effective action in the sigma model metric when the dilaton kinetic term is dropped.

$$S \sim \int d^2x \sqrt{-g} [e^{-2\phi} (R + \lambda - \frac{1}{4} F^2) + i\bar{\psi} \not{D} \psi] \quad (2.48)$$

By a conformal rescaling of the metric  $g_{\mu\nu} \rightarrow g_{\mu\nu} \exp(-2\phi)$  the action can be brought

to a form similar to the Einstein-Maxwell action with a modified space-time dependent coupling for the kinetic term of the electromagnetic field. The coupling involves the dilaton field  $\phi$ . The metric equations would give rise to in general a curved background and the problem really boils down to the study of Schwinger model in such a background. Actually string theory offers many more copies of the abelian gauge fields and the fermions can interact with all of them. For simplicity we have restricted us to this case.

Now as above we first bosonize the fermionic fields and the resulting action is exactly the same as (2.26) with an additional kinetic term for the abelian gauge field  $\sim -\exp(-2\phi)F^2$ . So as is observed from the above general analysis that the essential spectrum of the theory does not alter even after introducing an arbitrary curved background the conclusions made in [5] still hold. So at least it is clear that some quantum gravitational considerations are necessary to address the question of whether the information is really lost.

### 3 Temperature and entanglement entropy

There have been some attempts at calculating the entropy of quantum fields in black hole backgrounds [13], in contrast to the more conventional Bekenstein entropy [14]. The values thus obtained are contributions to the entropy of the black hole - field system. These calculations have produced divergences [15]. We shall see that similar phenomena occur in general for two dimensional black holes also [16] in a different way. Simple Einstein action is trivial in two dimensions and related to the Euler number of the underlying manifold by the Gauss-Bonnet theorem. The situation is slightly non-trivial in string theory where we have an extra scalar field, the dilaton, coupled with the curvature. The model can be extended to have electromagnetic interactions and fermions. Actually if

we consider e.g. the heterotic string on eight torus the resulting effective theory in two dimensions automatically has many copies of Abelian gauge fields and also have fermions. Many black hole solutions of this model have been found with non-zero charge.

An eternal black hole can generally be taken as

$$ds^2 = -g_{tt}(r) dt^2 + g_{rr}(r) dr^2 \quad (3.1)$$

together with a dilaton  $\varphi$ . A classical black hole has a horizon beyond which nothing can leak out. This suggests that it can be assigned a zero temperature. But the relation between the area of the horizon and the mass and other parameters like the charge indicates a close similarity [17] with the thermodynamical laws, thus allowing the definition of a temperature. This analogy was understood as being of quantum origin and made quantitative after the discovery of Hawking radiation [18]. The associated Hawking temperature vanishes only in the classical limit. The thermodynamics of black holes has been extensively studied since then.

Most of the studies were first made for the simplest kind of black hole, *viz.*, the Schwarzschild space-time. Of more recent interest is the case of the so-called extremal black holes which have peculiarities not always present in the corresponding non-extremal cases [19, 20]. For extremal Reissner - Nordstrom black holes, e.g., the naïvely defined temperature is zero, but the *area*, which is usually thought of as the entropy, is nonzero. For extremal dilatonic black holes, where the temperature is *not* zero, the area vanishes. As is well known that under some approximations two dimensional black holes appear in the extremal limit of some dilatonic black holes.

In this section we shall reexamine the temperature of a black hole [21]. We discuss the conical singularity approach in detail. The best known method of calculating the temperature of a black hole is through the relation with surface gravity. To distinguish



this temperature from those arising in other approaches, we may call it the Hawking temperature. Here

$$T = \frac{1}{2\pi\sqrt{g_{rr}}} \left. \frac{d\sqrt{-g_{tt}}}{dr} \right|_{r=r_h} \quad (3.2)$$

where,  $r = r_h$ , describes the horizon. However, there are other approaches to the temperature, and these must be considered in view of the peculiarities of extremal black holes.

First we consider the question of a conical singularity on passing to imaginary time. The metric

$$ds^2 = dr^2 + r^2 d\theta^2, \quad (3.3)$$

which describes the flat Euclidean metric in polar variables, can be supposed to describe distances on the surface of a cone. The cone has a singularity at its tip  $r = 0$ , except in the limiting case when the cone opens out as a plane. In this situation  $\theta$  has a periodicity  $2\pi$ , so one may say that the conical singularity is avoided by making  $\theta$  an angular variable with this period. This is relevant for black holes because such a singularity tends to arise in the Schwarzschild and in the non-extremal cases. In this approach, one passes to imaginary time and writes the metric as

$$\begin{aligned} ds^2 &= g_{tt}(r) dt^2 + g_{rr}(r) dr^2 \\ &= \Omega(\rho)(d\rho^2 + \rho^2 d\tau^2), \end{aligned} \quad (3.4)$$

where  $\tau = \alpha t$  with the constant  $\alpha$  so chosen as to make the conformal factor  $\Omega$  finite at the horizon. For consistency, one requires

$$\rho = e^{\alpha r_*}, \quad (3.5)$$

where  $r_*$  is defined by

$$dr_* = \sqrt{\frac{g_{rr}}{g_{tt}}} dr. \quad (3.6)$$

which implies that  $\rho$  vanishes at the horizon, i.e., as  $r_* \rightarrow \infty$  Now

$$\Omega = \frac{g_{tt}}{\alpha^2 \rho^2} \quad (3.7)$$

can be made finite at the horizon by making  $\rho^2$  vanish linearly as the horizon is approached i.e., by choosing  $\alpha$ . Now for the conical singularity to be avoided, one must have a periodicity of  $2\pi$  for  $\tau$ , i.e., a periodicity for  $t$  given by  $2\pi/\alpha$ . This corresponds to a temperature

$$T = \frac{\alpha}{2\pi} \quad (3.8)$$

which is the standard result. Thus for a general black hole, what may be called the *Unruh* temperature may/may not agree with the Hawking temperature. In four dimensions, in fact, there are many such extremal cases, where these things are very different. Let us now look for the expression of entanglement entropy of scalar field in the background of such a general black hole.

As argued in [22] the partition function for the system can be defined by the (Euclidean) Lagrangian path integral for the gravitational action coupled with matter fields. The dominant contribution will come from the classical solutions of the action. We may approximate the Euclidean action by taking something like

$$S_E[g, \varphi, A, \Phi] = S_1[g_{cl}, \varphi_{cl}, A_{cl}] + S_2[g_{cl}, \Phi] + \dots \quad (3.9)$$

where  $\Phi$  is the scalar field to be considered in the background of the dilatonic black hole and  $A$  stands for the background electromagnetic field. Quantum fluctuations of the

metric, the electromagnetic field and the dilatonic field are neglected and these variables are frozen to their classical values. The partition function can then be taken as

$$\mathcal{Z} = e^{-S_1[g_{cl}, \varphi_{cl}, A_{cl}]} \int \mathcal{D}\Phi e^{-S_2[g_{cl}, \Phi]}. \quad (3.10)$$

We come now to the contribution of the scalar field  $\Phi$  to the partition function. To calculate this we employ the brick-wall boundary condition [13]. In this model the field is cut off just outside the horizon. Mathematically,

$$\Phi(x) = 0 \quad \text{at } r = r_h + \epsilon \quad (3.11)$$

where  $\epsilon$  is a small, positive, quantity and signifies an ultraviolet cut-off. Let us set also an infrared cut-off (anticipating the result)

$$\Phi(x) = 0 \quad \text{at } r = \Lambda \quad (3.12)$$

with  $\Lambda \gg r_h$ . The wave equation for a scalar field in this space-time reads

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) - m^2 \Phi = 0. \quad (3.13)$$

A solution of the form

$$\Phi = e^{-iEt} f_E(r) \quad (3.14)$$

satisfies the radial equation

$$\frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} g^{rr} \partial_r f_E) + k_r^2 f_E = 0. \quad (3.15)$$

An  $r$ -dependent radial wave number can be introduced from this equation by

$$k_r(r, E) = [g^{tt} E^2 - m^2]^{1/2}. \quad (3.16)$$

Only such values of  $E$  are to be considered here that the above expression is nonnegative. The values are further restricted by the semi-classical quantization condition

$$n_r \pi = \int_{r_h+\epsilon}^{\Lambda} dr \sqrt{g_{rr}} k_r(r, E), \quad (3.17)$$

where  $n_r$  has to be a positive integer.

Accordingly, the free energy  $F$  at inverse temperature  $\beta$  is given by the formula

$$\begin{aligned} \beta F &= \sum_{n_r} \ln(1 - e^{-\beta E}) \\ &\approx \int dn_r \ln(1 - e^{-\beta E}) \\ &= - \int d(\beta E) (e^{\beta E} - 1)^{-1} n_r \\ &= - \frac{\beta}{\pi} \int_{r_h+\epsilon}^{\Lambda} dr \sqrt{g_{rr}} \int \frac{dE}{e^{\beta E} - 1} \sqrt{g^{tt} E^2 - m^2}. \end{aligned} \quad (3.18)$$

Here the limits of integration for  $E$  are such that the arguments of the square roots are nonnegative. The  $E$  integral can be evaluated only approximately where the lower limit of the  $E$  integral has been approximately set equal to zero. If the proper value is taken, there are corrections involving  $m^2 \beta^2$  which will be ignored here. The entanglement entropy can be obtained from the formula

$$S = \beta^2 \frac{\partial F}{\partial \beta}. \quad (3.19)$$

Let us now consider a particular two dimensional black hole. We consider the low energy effective action of heterotic string on eight torus. In that action if we set all the gauge fields and moduli fields to zero it takes the following form

$$\mathcal{S} \sim \int d^2 x \sqrt{-g} e^{-2\varphi} [R + 4(\nabla\varphi)^2 + 4\lambda^2] \quad (3.20)$$

where,  $1/\lambda$  is a length scale which the action inherits from higher dimensions. This action is known to have black hole solution [23]. In fact, this black hole is the limiting eternal

black hole solution of the more interesting dynamical ones, which were obtained in [24]. However, here we shall be talking only about the eternal solution

$$ds^2 = -e^{-2\zeta} dx dy. \quad (3.21)$$

$x, y$  represent the Kruskal-like coordinates if the solution is compared with the ‘mock Schwarzschild’ metric. In that case the Schwarzschild-like coordinates can be introduced via the transformations

$$\begin{aligned} \lambda x &= e^{\lambda\sigma^+} \\ \lambda y &= -e^{-\lambda\sigma^-} \end{aligned} \quad (3.22)$$

where,  $\sigma^\pm = t \pm \sigma$ , are the light cone coordinates. The coordinate  $\sigma$  is like the Schwarzschild - radial coordinate. The conformal factor in front of the metric is given by  $\exp(2\zeta) = -\lambda^2 xy + M/\lambda$ . The horizon of this black hole is at  $y = 0$ , the curvature singularity is at the space-like curve  $xy = M/\lambda^3$  and the asymptotic region is described by  $\sigma = \infty$ .  $M$  represents the mass of the black hole. To have a more familiar form let us make this coordinate redefinition

$$1 - 2M/r = (1 + Me^{-2\lambda\sigma}/\lambda)^{-1}. \quad (3.23)$$

This puts the metric in the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{4\lambda^2 r^2} \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \quad (3.24)$$

The horizon is mapped to  $r = 2M$ , the curvature singularity is at  $r = 0$  and the asymptotic regions are described by  $r = \infty$ . This form is very suggestive to compare the solution with the ‘mock Schwarzschild’ metric. Now let us try to estimate its temperature and entanglement entropy following the general procedure described above. First of all the

Unruh temperature can be fixed easily. the variable  $\rho$  introduced above is  $\rho = \exp(\alpha\sigma) = (r - 2M)^{\alpha/2M}$ . So  $\Omega = (1 - 2M/r)/\alpha^2\rho^2$  is finite at the horizon provided we set  $\alpha = \lambda$ . This in turn implies  $T = \lambda/2\pi$ . Fortunately, for this solution the Hawking temperature agrees with the Unruh temperature and the conical singularity can be removed by this choice of the angular periodicity. Using the general formula above we can also calculate the entanglement entropy for this black hole. The free energy is given by

$$F \sim -\frac{1}{\lambda\beta^4} \ln \frac{\Lambda}{\epsilon}$$

where  $\beta = 2\pi/\lambda$ . We neglected the other proportionality factors in the expression of free energy. So the entropy is  $S \sim \lambda^2 \ln(\Lambda/\epsilon)$ , it receives divergences from the ultraviolet as well as from the infrared regions [25].

There is another way to fix the temperature of a black hole which is from the study of Hawking radiation. For this we have to calculate the expectation value of the number operator of say, a scalar field in the Schwarzschild-like coordinate system in the vacuum of the Kruskal-like observer. The celebrated result is that it would come out to be a Bose distribution which is interpreted as a thermal radiation coming out of the black hole. The distribution function then corresponds to a definite temperature. As we have already mapped the black hole to a ‘Mock Schwarzschild’-like form (at least near the horizon  $r \approx 2M$  there is hardly any difference) the space-time can also be mapped to a Rindler one. Essentially, the Kruskal construction given above is nothing but the familiar transformations between the Rindler and the flat spaces near the horizon. To see this explicitly let us scale the time  $t \rightarrow t/\lambda$ , and define the Rindler coordinate  $\eta = \sqrt{1/\lambda M} \exp(\lambda\sigma)$ . Then the line element near the horizon takes the standard form

$$ds^2 \approx -\frac{\lambda}{M} dx dy = -\eta^2 d\tau^2 + d\eta^2 \quad (3.25)$$

where  $\tau = \lambda t$ . As it is well known that the Rindler observer sees a thermal bath of temperature  $1/2\pi$  and we have scaled the time coordinate by  $\lambda$  the actual temperature which the asymptotic observer sees is  $\lambda/2\pi$  consistent with the earlier observations.

Finally it is important to see that why the entanglement entropy is so important in the study of black holes and its semi-classical properties. We can very simply demonstrate that the entanglement between the different partitions in the space-time leads us to the consideration of a thermal picture. The Hawking radiation is an outcome of such a partitioning. As we have seen that the space-time near to the horizon is essentially flat and can be mapped to a Rindler one, the physics of that region is captured by the transformations between these two frames. We shall be calling these two observers as the ‘flat’ and the ‘Rindler’ observers accordingly. As the flat observer does not have any coordinate singularity at the horizon she has access to both sides of the horizon. However, the Rindler observer doesn’t have access across the horizon. So from the point of view of the flat observer the Rindler space-time is only a part of the entire space-time and the Rindler observer has access only to that part. We shall see explicitly the effect of such a splitting. If we take the flat space coordinate as  $X, Y$  with  $x = T + X$  and  $Y = T - X$  then the Rindler mappings are

$$\begin{aligned} T &= \sqrt{\frac{M}{\lambda}} \eta \sinh \tau \\ X &= \sqrt{\frac{M}{\lambda}} \eta \cosh \tau. \end{aligned} \tag{3.26}$$

Then a small translation of the Rindler time  $\delta\tau = \epsilon$  with  $\delta\eta = 0$  corresponds to the following flat space-time transformations

$$\begin{aligned} \delta T &= X\epsilon \\ \delta X &= T\epsilon. \end{aligned} \tag{3.27}$$

It is in fact, the Lorentz boost along the space direction  $X$ . The generator is

$$H_R = \int_{-\infty}^{+\infty} dX (\mathcal{H}X - \mathcal{P}T) \quad (3.28)$$

where  $\mathcal{H}, \mathcal{P}$  are respectively the Hamiltonian and momentum operators in the flat space-time. Now the generator being a conserved operator in time the integral can be evaluated at any time slice. Let us set it at  $T = 0$ . Then it takes the simple looking form  $H_R = \int dX X \mathcal{H}(X, 0)$ . Now if we look only along the axis  $T = 0$  the space of the Rindler observer is split into the two parts  $X > 0$  and  $X < 0$ . So we split the generator also accordingly as  $H_R = H_> - H_<$ , where

$$\begin{aligned} H_> &= \int_{-\infty}^{+\infty} dX \theta(+X) X \mathcal{H} \\ H_< &= - \int_{-\infty}^{+\infty} dX \theta(-X) X \mathcal{H}. \end{aligned} \quad (3.29)$$

Now given the mode expansions of a scalar field of mass  $m$  in terms of a set of creation and annihilation operators in the flat space-time it is possible to find out another set through the generator  $H_R$  can be expressed as  $H_R = \int d\omega \omega a_\omega^\dagger a_\omega$ . In fact, given the free field expansion as

$$\Phi(X, T) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{4\pi k_0}} [b_k \exp(ikX - ik_0T) + \text{h.c.}] \quad (3.30)$$

with  $k_0^2 = m^2 + k^2$  and  $[b_k, b_{k'}^\dagger] = \delta_{kk'}$  the construction of  $a_\omega$  is given in [26]

$$b_k = \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi k_0}} a_\omega \exp\left(-i\omega \ln \frac{k_0 + k}{m}\right). \quad (3.31)$$

This construction gives  $[a_\omega, a_{\omega'}^\dagger] = \delta_{\omega\omega'}$ . Now the general problem of splitting is as follows: let the annihilation operators corresponding to the Hamiltonians  $H_>$  and  $H_<$  are  $a_>$  and  $a_<$  respectively. Let us also impose the condition that these two regions  $>$  and  $<$  are



completely disjoint and there is no correlation between them. So we have the following set of conditions

$$\begin{aligned}
[a_{>}(\omega), a_{<}(\omega')] &= 0 \\
[a_{>}(\omega), a_{<}^\dagger(\omega')] &= 0 \\
[a_{>}(\omega), a_{>}^\dagger(\omega')] &= \delta_{\omega\omega'} \\
[a_{<}(\omega), a_{<}^\dagger(\omega')] &= \delta_{\omega\omega'} \\
\int_{-\infty}^{+\infty} d\omega \, \omega \, a_\omega^\dagger a_\omega &= \int_{-\infty}^{+\infty} d\omega \, \omega \left( a_{>}^\dagger(\omega) a_{>}(\omega) - a_{<}^\dagger(\omega) a_{<}(\omega) \right). \quad (3.32)
\end{aligned}$$

If now the operators  $a_{>}$  and  $a_{<}$  are related to the flat space operators  $a_\omega$  and  $a_\omega^\dagger$  as follows

$$\begin{aligned}
a_{>}(\omega) &= \sum_{\omega'} A_{\omega\omega'} a_{\omega'} + B_{\omega\omega'} a_{\omega'}^\dagger \\
a_{<}(\omega) &= \sum_{\omega'} C_{\omega\omega'} a_{\omega'} + D_{\omega\omega'} a_{\omega'}^\dagger \quad (3.33)
\end{aligned}$$

we can try to solve for these coefficients from the above conditions. Obviously the general solution can not be obtained, but we can look for at least a special solution. A one parameter special solution is

$$\begin{aligned}
A_{\omega\omega'} &= \delta_{\omega\omega'} \frac{1}{\sqrt{1 - e^{-\beta\omega}}} \quad , \quad B_{\omega\omega'} = \delta_{\omega, -\omega'} \frac{e^{-\beta\omega/2}}{\sqrt{1 - e^{-\beta\omega}}} \\
C_{\omega\omega'} &= \delta_{\omega, -\omega'} \frac{1}{\sqrt{1 - e^{-\beta\omega}}} \quad , \quad D_{\omega\omega'} = \delta_{\omega\omega'} \frac{e^{-\beta\omega/2}}{\sqrt{1 - e^{-\beta\omega}}}. \quad (3.34)
\end{aligned}$$

This solution corresponds to Hawking radiation. To see this explicitly let us define the vacuum for the flat observer as  $b_k|0\rangle = a_\omega|0\rangle = 0$ . In this vacuum one of the Rindler observer, say in the region  $>$ , would see a spectrum as

$$\langle 0 | a_{>}^\dagger(\omega) a_{>}(\omega') | 0 \rangle = \sum_{\lambda} B_{\omega\lambda}^* B_{\omega'\lambda}, \quad (3.35)$$

which for the special solution boils down to the form a thermal spectrum obeying Bose-statistics

$$\sum_{\lambda} B_{\omega\lambda}^* B_{\omega'\lambda} = \delta_{\omega\omega'} \frac{1}{e^{\beta\omega} - 1}. \quad (3.36)$$

Here the parameter  $\beta$  should be identified with the temperature. As from the other considerations the temperature is known to be  $\lambda/2\pi$  the Hawking spectrum is explicitly known in this case.

## 4 Discussion

We see in this paper that QED<sub>2</sub> in non-trivial backgrounds does not show any sign of being in a different phase other than those in the flat cases. It is still hard to give a completely general proof of this in an arbitrary background but we could establish the fact at least in some black hole background. Possibly explicit calculations of Green functions could be possible in this case but it is still hard to draw conclusions even from that as we have experienced already in the flat case. As a possible application of these solutions we considered the fermion- black hole scattering problem. The above conclusion have an important implication on this, however. We could draw conclusions that the problem of information loss for fermions can not be solved if we treat gravity classically and this motivates further the search for a viable quantum theory of gravity to avoid such unpleasant events around these special backgrounds.

We also did analysis about the nature of QED<sub>2</sub> in this special background. As the spectrum of the abelian theory contains only a scalar field the analysis is sufficiently simplified. We showed that why people are so much interested about the entropy coming from the splitting of fields. The essential reason is that the Hawking's calculations are also

entirely based on such splittings of space-times. The field theoretic calculations of entropy, however, show divergences coming both from the ultraviolet and the infrared regions. Many string theoretic calculations have been done afterwards and they're potentially finite [27]. However, it has to be remembered that the calculations involving such splittings are associated with scales. e.g. if the characteristic length scale of one of the regions gets too small compared to that of the other, then essentially there is no splitting and there should be hardly any thermal spectrum seen by the Rindler observer. In that case we expect that the temperature should go to infinity (to have a vanishing spectrum) as the region inside the horizon gradually shrinks to smaller and smaller sizes by emitting Hawking radiations. Actually somewhere, the field theoretic framework possibly breaks down and we need a more microscopic theory such as string theory to probe. The divergences are possible indications of that failure of field theory to handle the high energy regimes if and only if it becomes impossible to accommodate those divergences in some reasonable renormalization scheme, which is also far from our sight essentially in higher dimensions. Also the fact that the temperature goes to infinity as we go towards the end point of Hawking radiation indicates that somewhere in between the thermal description should break down as well. We do not yet have a good picture of either of these two problems [28] and hopefully in the coming years we will have a unifying point of view [29, 30] of resolving both these problems of information loss and black hole thermodynamics simultaneously.

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